A SECOND-ORDER SEMI-LAGRANGIAN INTEGRATION SCHEME FOR INCOMPRESSIBLE FLOWS

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ABSTRACT. In this paper we present a scheme based on semi-Lagrangian integration (SLI) for approximating the solution of the Navier-Stokes equations. Exact solutions of these equations are generally difficult to find particularly when strict physical conditions are required to hold. However, in this paper we focus on the stability of an approximate solution. The solution presented is unconditionally stable and therefore allows for arbitrary time steps. It can be used for a stable and an efficient real-time simulation of fluid flow. Applications are found in computer graphics, weather forecasting and other areas that require real-time simulation of fluid flow.

1. INTRODUCTION

The Navier-Stokes equations for incompressible flow model the dynamics of fluid flow in space $\mathbf{x} = (x, y, z)$ and time t. The fluid is represented by its velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$ fields. The non-dimensional form of the Navier-Stokes equations is given by:

$$(1.1) \qquad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

where $\mathbf{u}(\mathbf{x}, t)$ is a vector velocity field, $p(\mathbf{x}, t)$ is a scalar pressure field, $\mathbf{f}$ is a constant external force acting on the fluid element, and μ is the coefficient of viscosity of the medium.

The basic assumption that governs the Navier-Stokes equations is that the fluid is a continuum; that is, it is not composed of discrete particles. Also, it is assumed that all the fields of interest such as pressure, velocity, density, temperature, etc, are differentiable.

Note that incompressible flow (isochoric flow) refers to a flow in which the material density is constant within an infinitesimal volume that moves with the velocity of the fluid. Put differently a flow is incompressible if the divergence of the fluid velocity is zero. Therefore incompressible flow does not imply that the fluid itself is incompressible.

2. SEMI-LAGRANGIAN INTEGRATION METHODS

2.1. **Background.** In general, there are two major approaches to the mathematical description of fluid flows namely, Eulerian and Lagrangian approaches. In Eulerian approach, there is a single fixed reference frame from which the flow is observed. This may be likened to an observer standing on a river bank watching the flow evolve. On the other hand, in the Lagrangian viewpoint, the reference frame is

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not fixed but moves with the flow. Thus, the reference frame in the Lagrangian viewpoint is dependent on both space and time. These two approaches give rise to two distinct numerical models.

It is the Lagrangian approach that we use in this paper in determining an unconditionally stable solution to the Navier-Stokes equations.

2.2. Mathematical Foundation of Semi-Lagrangian Methods. Consider the convection equation

$$(2.1) \qquad \frac{\partial \phi}{\partial t} + \mathbf{V}(x,t).\nabla \phi = 0$$

where ϕ is a scalar field and $\mathbf{V}(x,t)$ is a velocity function. Clearly (2.1) advects ϕ through the velocity field $\mathbf{V}$. Semi-Lagrangian methods arise from the observation that (2.1) propagates ϕ along characteristic curves $x = \chi(t)$ defined by the equation

$$(2.2) \qquad \dot{\chi}(t) = \mathbf{V}(\chi(t),t), \quad \chi(0) = x_0.$$

Therefore we can find values of ϕ at any time t simply by finding the characteristic curve that passes through (x,t) following it backward to some previous point (x_0, t_0) where the value of ϕ is known, and then setting $\phi(x,t) = \phi(x_0, t_0)$.

Thus, (2.1) is decoupled into a system of ordinary differential equations as follows:

$$(2.3) \qquad \begin{cases} \frac{d\chi(t)}{dt} = \mathbf{V}(x,t) \\ \frac{d\phi}{dt} = 0. \end{cases}$$

In the following sections, we shall take a closer look of advection using semi-Langragian methods. This, we hope, will give us a firmer understanding of the underpinnings of the Lagrangian methods.

2.3. Linear Advection. The general advection problem is in the form

$$(2.4) \qquad \begin{cases} \frac{\partial u}{\partial t} + a(x,t,u)\frac{\partial u}{\partial x} = g(x,t,u) \\ u(x,0,u) = u_0(x,u). \end{cases}$$

We shall develop a semi-Lagrangian integration scheme for (2.4). The construction of the scheme is in three steps, namely:

1. Trajectory (characteristic) tracing
2. Interpolation
3. Time discretization.

These steps constitute the three components of semi-Lagrangian scheme for advection.

2.3.1. *Homogeneous Constant Velocity Advection.* Suppose in this case that the source term $g(x,t,u) = 0$ and $a(x,t,u) = $ constant. Thus (2.4) reduces to the form

$$(2.5) \qquad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0.$$

Using the method of characteristics, (2.5) is rewritten as

$$(2.6) \qquad \frac{d\chi}{dt} \quad = \quad a$$

$$(2.7) \qquad \frac{du}{dt} \quad = \quad 0.$$

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Equations (2.6) and (2.7) are to be solved for the functions $x(t)$ and $u(x(t),t)$, respectively. Indeed, (2.6) describes the trajectory of an object moving at the advecting velocity a. Given the initial condition $x = x_0$ at $t = 0$, we have

$$(2.8) \qquad x(t) = at + x_0, \quad t > 0.$$

The solution of (2.7) is

$$(2.9) \qquad u(x,t) = \text{constant}.$$

This means that $u(x,t)$ is constant along a characteristic curve. Combining (2.8) and (2.9) we obtain

$$(2.10) \qquad u(x_0 + at, t) = u(x_0, t), \quad t > 0.$$

Therefore, the solution of (2.5) is

$$(2.11) \qquad u(x,t) = u_0(x - at).$$

Using the above description to obtain a numerical scheme for solving (2.5) results in a **semi-Lagrangian** method for solving the advection problem (2.5).

Characteristic Tracing and Interpolation

Implementation of semi-Lagrangian integration scheme for the advection problem involves using a finite difference representation of the solution of (2.5). Suppose that the grid has space and time mesh Δx and Δt, respectively. Then u_j^n denotes the approximation of the solution $u(x_j, t_n)$, where $x_j = j\Delta x$ and $t_n = n\Delta t$.
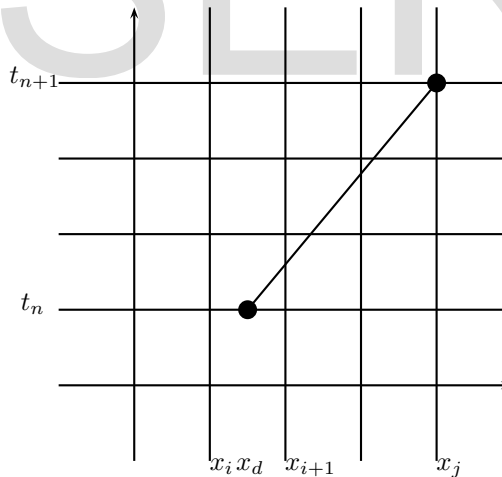

FIGURE 1. Computational Grid

A key assumption in constructing the semi-Lagrangian scheme is that the solution is known at time $t = t_n$. Thus the method of characteristics is used to advance the solution by one time step. In other words, given $u(x_j, t_n)$, we desire to find the finite difference solution $u_j^{t_n+1}$ at time $t = t_{n+1}$ for each j. In Figure 1, we show the characteristic which passes through the node $x = x_j$ at time $t = t_{n+1}$. This characteristic is a straight line with slope $1/a$. Tracing back in time, it is seen that it passes through the point $x = x_d$ at time $t = t_n$. Specifically, x_d is the departure

point of the characteirstic at time $t = t_n$ which arrives at $x = x_j$ in time $t = t_{n+1}$.

Next, in the general advection case, time discretization of the solution is performed at this stage of the scheme. Here, we consider the solution u which has been shown to be constant along any characteristic. Therefore, we have

$$(2.12) \qquad u_j^{n+1} = u_d^n,$$

where u_d^n denotes the value of u at the departure point.

Here a problem involving x_d arises. The problem is that, in general, x_d is not a grid point. For if it happens to be a grid point, $x_d = x_i$, for example, then (2.12) would give

$$(2.13) \qquad u_j^{n+1} = u_i^n.$$

Otherwise, one would have to interpolate to obtain an approximation for u_j^{n+1}. A variety of interpolation schemes may be considered for use. Suppose that x_d lies between the nodes x_i and x_{i+1}, then a linear interpolation yields

$$(2.14) \qquad u_d^n = \frac{x_d - x_i}{x_{i+1} - x_i} u_{i+1}^n + \frac{-x_d + x_{i+1}}{x_{i+1} - x_i} u_i^n.$$

It then follows that u_d^n is an approximation of $u(x_d, t_n)$.

Equation(2.12) provides a finite difference approximation at time t_{n+1} from the information provided at t_n. Clearly, this scheme has only first order accuracy. One can easily derive a higher order scheme by using an interpolation scheme of higher order.

It is worth mentioning that the accuracy of the charateristic tracing determines the efficiency of the semi-Lagrangian integration method.

2.3.2. *Variable Velocity Advection.* In this section, we consider (2.4) again but now with a variable velocity field $a(x, t, u)$ while $g(x, t, u)$ remains zero. In this case, a numerical scheme is required to determine the departure point x_d.

We want to solve

$$(2.15) \qquad \frac{d\chi}{dt} = a(x, t, u)$$

$$(2.16) \qquad \frac{du}{dt} = 0.$$

First, we solve

$$\frac{dx}{dt} = a(x, t, u)$$

for $x(t_n)$ subject to the condition $x(t_{n+1}) = x_j$. The point $x(t_n)$ is in this case the departure point for x_j. To numerically solve the characteristic equation we discretize the equation. The midpoint rule gives reasonably accurate results for solving this problem. Thus,

$$(2.17) \qquad \frac{x_j - x_d}{\Delta t} = a(x_m, t_n + \Delta t/2)$$

where $x_m = \frac{1}{2}(x_j + x_d)$. Clearly, (2.17) is an implicit equation and can be solved by fixed point iteration.

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Setting

$$c = x_j - x_d,$$

then (2.17) may be expressed as

$$(2.18) \qquad c = \Delta ta(x_j - c/2, t_n + \Delta t/2).$$

Consequently, the iteration

$$(2.19) \qquad c^{(k+1)} = \Delta ta(x_j - \frac{1}{2}c^{(k)}, t_n + \frac{1}{2}\Delta t)$$

gives a numerical solution of (2.17) and we obtain

$$(2.20) \qquad x_d = x_j - c.$$

Observe that $t = 0, t_1, t_2, \ldots, t_n$. Therefore, to obtain a value at $t_{n+1/2} = t_n + \frac{1}{2}\Delta t$, some extrapolation is required.

Taylor series expansion can be used to obtain extrapolation formulas that are reasonably good. For example, a second order extrapolation formula is

$$a^{n+\frac{1}{2}} = \frac{1}{2}(3a^n - a^{n-1}) + O(\Delta t^2).$$

2.3.3. *Nonzero Source Term.* Now we consider the case where the source term is nonzero, that is, $g(x, t, u) \neq 0$ in (2.4). The system reduces to the form

$$(2.21) \qquad \frac{d\chi}{dt} = a(x, t, u)$$

$$(2.22) \qquad \frac{du}{dt} = g(x, t, u).$$

This leads us to the final step of the SLI scheme. This involves time discretization of (2.22).

Time Discretization

We have

$$(2.23) \qquad \frac{u_j^{n+1} - u_d^n}{\Delta t} = (1 - \alpha)g_d^n + \alpha g_j^{n+1}$$

where $u_d^n \approx u(x_d, t_n)$ and $g_d^n \approx g(x_d, t_n)$. The parameter α is an implicitness parameter. The procedure is fully implicit if $\alpha = 1$ and explicit if $\alpha = 0$. Otherwise, the procedure is considered semi-implicit.

3. Solution of the Navier-Stokes

Having established the fundamental concepts of semi-Lagrangian integration, we can now apply the scheme to the Navier-Stokes equation (1.1):

$$(3.1) \qquad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

By doing so, we obtain

$$(3.2) \qquad \begin{cases} \frac{d\chi}{dt} = \mathbf{u}(\mathbf{x}(t), t) \\ \frac{d\mathbf{u}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \end{cases}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u}.\nabla).$$

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We discretize using a backward differentiation formula to obtain the semi-Lagrangian scheme:

$$(3.3) \qquad \frac{1}{\Delta t}\left(\frac{3}{2}\mathbf{u}^{n+1} - 2\mathbf{u}_d^n + \frac{1}{2}\mathbf{u}_d^{n-1}\right) = -\nabla p^{n+1} + \mu\Delta\mathbf{u}^{n+1} + \mathbf{f}^{n+1}.$$

Equation (3.3) is solved using the *Projection Method.*

3.1. **The Projection Method.** The projection method for solving the flow equation is based on the observation that in the equation

$$(3.4) \qquad \mathbf{u}_t + (\mathbf{u}.\nabla)\mathbf{u} + \nabla p \;\; = \;\; \mu\nabla^2\mathbf{u},$$

$$(3.5) \qquad \nabla.\mathbf{u} \;\; = \;\; 0,$$

$\mathbf{u}_t + \nabla u$ is a Helmholtz-Hodge decomposition. Essentially, Helmholtz-Hodge decomposition says that any vector function $\mathbf{V}(x)$ can be decomposed into a divergence-free part $\mathbf{u}$ and the gradient of a scalar potential ϕ, that is,

$$(3.6) \qquad \mathbf{V}(x) = \mathbf{u}(x) + \nabla\phi(x)$$

with $\nabla.\mathbf{u} = 0$, and $< \mathbf{u}, \nabla\phi >= 0$ for a suitably chosen inner product. This implies that $\mathbf{u}$ and $\nabla\phi$ are orthogonal.

Therefore, the projection of $\mathbf{V}$ is

$$(3.7) \qquad \mathbf{u} = P(\mathbf{V}).$$

The procedure begins by approximating (3.4)

$$(3.8) \qquad \mathbf{u}_t^* + (\mathbf{u}.\nabla)\mathbf{u} + \nabla q = \mu\nabla^2\mathbf{u}^*,$$

while the divergence-free velocity is computed using

$$(3.9) \qquad \mathbf{u}(t) = P(\mathbf{u}^*(t))$$

where $\nabla q(x,t)$ approximates the pressure gradient. Now

$$(3.10) \qquad \mathbf{u}^* = \mathbf{u} + \nabla\phi$$

implies that

$$(3.11) \qquad \nabla.\mathbf{u}^* = \nabla.(\mathbf{u} + \nabla\phi) = \underbrace{\nabla.\mathbf{u}}_{0} + \nabla^2\phi.$$

It then follows that

$$(3.12) \qquad \nabla^2\phi = \nabla.\mathbf{u}^*$$

which is an elliptic constraint. The pressure can be recovered using

$$(3.13) \qquad \nabla p = \nabla(q + \phi_t) - \mu\nabla^2\nabla\phi.$$

Substituting (3.10) into (3.8) yields

$$\begin{aligned}(\mathbf{u} + \nabla\phi)_t + (\mathbf{u}.\nabla)\mathbf{u} + \nabla q &= \mu\nabla^2(\mathbf{u} + \nabla\phi) \\ \mathbf{u}_t + (\nabla\phi)_t + (\mathbf{u}.\nabla)\mathbf{u} + \nabla q &= \mu\nabla^2\mathbf{u} + \mu\nabla^2\nabla\phi)\end{aligned}$$

That is

$$(3.14) \qquad \mathbf{u}_t + (\mathbf{u}.\nabla)\mathbf{u} + \left\{\nabla(q + \phi_t) - \mu\nabla^2\nabla\phi\right\} = \nu\nabla^2\mathbf{u}.$$

Comparing (3.4) and (3.14) we obtain (3.13).

Implementation of the projection method differs depending on how the advection term and the quantity q are approximated. We also have to consider whether (3.8)

is advanced explicitly or implicitly and how the pressure gradient update formula (3.13) is approximated.

We now summarize the projection method for implementation purposes. The projection method is a three-step procedure.

Step 1: Here we split (3.3) by ignoring the pressure term. Thus, we have

$$(3.15) \qquad \frac{1}{\Delta t}\left(\frac{3}{2}\mathbf{u}^* - 2\mathbf{u}_d^n + \frac{1}{2}\mathbf{u}_d^{n-1}\right) = \mu\Delta\mathbf{u}^* + \mathbf{f}^{n+1}.$$

Thus, given the velocity field $\mathbf{u}^n$ at time t^n, we solve (3.15) for the $\mathbf{u}^*$.

Equation (3.15) may be rewritten as

$$(3.16) \qquad (\frac{3}{2}I - \Delta t\mu\nabla^2)u^* = 2u_d^n - \frac{1}{2}u_d^{n-1} + \Delta t f^{n+1}.$$

Partitioning the domain into an $N \times N$ grid, we have

$$(3.17) \qquad \frac{3}{2}u_i^* - \frac{\Delta t\mu}{\Delta x}\left(u_{i+1}^* - 2u_i^* + u_{i-1}^*\right) = 2u_{d_i}^n - \frac{1}{2}u_{d_i}^{n-1} + \Delta t f_i^{n+1},$$

where $i = 1, 2, \ldots, N - 1$. In matrix form, the linear system appears as follows

$$(3.18) \qquad \mathbf{Au}^* = \mathbf{c},$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{3}{2}+2\alpha & -\alpha & 0 & 0 & 0 & \cdots \\ -\alpha & \frac{3}{2}+2\alpha & -\alpha & 0 & 0 & \cdots \\ 0 & -\alpha & \frac{3}{2}+2\alpha & -\alpha & 0 & \cdots \\ \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 & \frac{3}{2}+2\alpha & -\alpha \\ 0 & \cdots & \cdots & 0 & -\alpha & \frac{3}{2}+2\alpha \end{bmatrix}, \quad \mathbf{u}^* = \begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_{N-2}^* \\ u_{N-1}^* \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} \alpha u_0^* + 2u_{d_1}^n - \frac{1}{2}u_{d_1}^{n-1} + \Delta t f_1^{n+1} \\ 2u_{d_2}^n - \frac{1}{2}u_{d_2}^{n-1} + \Delta t f_2^{n+1} \\ \vdots \\ \alpha u_N^* + 2u_{d_{N-1}}^n - \frac{1}{2}u_{d_{N-1}}^{n-1} + \Delta t f_{N-1}^{n+1} \end{bmatrix} \quad \text{and} \quad \alpha = \frac{\Delta t\mu}{(\Delta x)^2}.$$

We see that since $\alpha > 0$, the coefficient matrix $\mathbf{A}$ is positive definite, symmetric, strictly diagonally dominant and tridiagonal. Therefore the Crout Factorization procedure or the Successive Over-relaxation (SOR) procedure can be used to solve the system. Details of these procedures can be obtained from any standard text on numerical analysis, for example [3],[1],[6].

Step 2: The solution, of course, must satisfy the incompressibility condition $\nabla.\mathbf{u}^{n+1} = 0$. This condition is addressed in this step by introducing a potential function ϕ^{n+1} where ϕ solves the Poisson equation:

$$(3.19) \qquad \Delta\phi^{n+1} = \frac{1}{\Delta t}\left(\nabla.\mathbf{u}^*\right).$$

Step 3: Finally, we solve

$$(3.20) \qquad \nabla p^{n+1} = \frac{3}{2}\nabla\phi^{n+1} - \Delta t\mu\nabla^3\phi^{n+1}.$$

for the pressure.

Steps two and three are not difficult to solve as there is a variety of approaches for solving such linear systems.

3.2. **Consistency.** A numerical discretization is said to be consistent if it tends to the associated differential equation as $\Delta x \to 0$ and $\Delta t \to 0$. Consistency is a necessary condition for approximate solution.

Taylor's series expansion of the terms of the numerical discretization can be used to verify consistency. The Taylor's expansions of the terms are substituted into the discretized equation and the resulting equation is checked if it tends to the differential equation as Δx and Δt approach zero. To demonstrate, consider the discretisation given in (3.3).

$$(3.21) \qquad u_i^n = u_i^{n+1} - \Delta t \left(\frac{du}{dt} \right)_i^{n+1} + \frac{(\Delta t)^2}{2} \left(\frac{d^2 u}{dt^2} \right)_i^n + \dots$$

$$(3.22) \qquad u_i^{n-1} = u_i^{n+1} - 2\Delta t \left(\frac{du}{dt} \right)_i^{n+1} + \frac{(2\Delta t)^2}{2} \left(\frac{d^2 u}{dt^2} \right)_i^{n+1} + \dots$$

Substituting (3.21) and (3.22) in (3.3) we obtain

$$(3.23) \quad \frac{1}{\Delta t} \left(\frac{3}{2} u_i^{n+1} - 2(u_i^{n+1} - \Delta t \left(\frac{du}{dt} \right)_i^{n+1} + \frac{(\Delta t)^2}{2} \left(\frac{d^2 u}{dt^2} \right)_i^n + \dots) + \right.$$

$$(3.24) \qquad \left. \frac{1}{2}(u_i^{n+1} - 2\Delta t \left(\frac{du}{dt} \right)_i^{n+1} + \frac{(2\Delta t)^2}{2} \left(\frac{d^2 u}{dt^2} \right)_i^{n+1} + \dots) \right)$$

$$(3.25) \qquad\qquad\qquad = -\nabla p^{n+1} + \nu \nabla^2 u_i^{n+1} + f_i^{n+1}$$

Simplifying we obtain

$$(3.26) \qquad \left(\frac{du}{dt} \right)_i^{n+1} + O((\Delta x)^2, (\Delta t)^2) = -\nabla p_i^{n+1} + \nu \nabla^2 u_i^{n+1} + f_i^{n+1}$$

Clearly the numerical discretization (3.3) is consistent since (3.26) tends to the original differential equation as $\Delta x \to 0$ and $\Delta t \to 0$.

3.3. **Stability.** Stability is the tendency of any perturbtions of the discretized system to decay. Therefore a system is unstable if an initial perturbation eventually becomes unbounded.

The eigenvalues of the coefficient matrix $\mathbf{A}$ are

$$\lambda_i = \frac{3}{2} + 4\alpha \left[\sin \left(\frac{\pi i}{2N} \right) \right]^2, \text{ for } i = 1, 2, \dots, N-1.$$

Since $\alpha = \frac{\Delta t \mu}{\Delta x} > 0$, it follows that $\lambda_i > \frac{3}{2}$, for all $i = 1, 2, \dots, N-1$. Since none of the eigenvalues of $\mathbf{A}$ are zero, it follows that A^{-1} exists. Therefore an error $\mathbf{e}^{(0)}$ in the initial data results in an error of $(\mathbf{A}^{-1})^n \mathbf{e}^{(0)}$ at the nth step. Since the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$, it follows that the spectral radius $\rho(\mathbf{A}^{-1})$ is bounded above by 1. This in turn implies that any perturbations to the discretized system eventually decay regardless of the choice of $\alpha = \frac{\Delta t \nu}{\Delta x}$. This establishes that our scheme is *unconditionally stable*.

The preceding discussion is summarized in the following statement.

Theorem 3.1. *Suppose the conditions of Equation (1.1) hold, then the semi-Lagrangian Integration scheme (3.2), (3.3) is unconditionally stable.*

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4. Conclusions

In this paper we have presented an unconditionally stable second-order scheme for solving the incompressible Navier-Stokes equations. Semi-Lagrangian integration method is used to update the momentum equation and stiffly backward difference procedure is used to treat the diffusion term. The desired quantities are sampled at the grid nodes of the domain. In theorem 3.1, we presented criteria for unconditional stability of the scheme.

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